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Bäcklund transformation of the noncommutative Gelfand-Dickey hierarchy

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Abstract: We study the Bäcklund transformation of the noncommutative Gelfand-Dickey(ncGD) hierarchy. By factorizing its Lax operator into the multiplication form of first order differential operator, the noncommutative modified KdV(ncMKdV) hierarchy and the Miura transformations are defined. Our results show that the ncMKdV equations are invariant under the cyclic permutation, and hence induces the Bäcklund transformation of the ncGD hierarchy.

Keywords: Integrable Hierarchies, Integrable Equations in Physics.

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Contents

1. Introduction

Since it was introduced by I.M. Gelfand and L.A. Dickey [1], the Gelfand-Dickey(GD) hierarchy (or say n-KdV hierarchy) has attracted a lot of attention in the research of integrable systems. The procedure for Hamiltonian theory of GD hierarchy as developed by M. Adler [2], I.M. Gefland and L.A. Dickey [1] becomes an efficient approach to researching the integrable system based on the (Pseudo-)differential operator in terms of Lax pair, see Dicky's book [3] for detail. In particular, the characteristics of GD hierarchy, such as soliton solution, τ function, symmetry and additional symmetry, Bäcklund transformation, gauge transformations, etc., are discussed extensively in references.

There have been several generalizations of GD hierarchy in recent years. For example, matrix GD hierarchy [4],constraint KP hierarchy [5], supersymmetric GD hierarchy [6] and supersymmetric KP hierarchy [7], q-GD hierarchy [8] and q-KP hierarchy [9]. All generalizations for GD hierarchy have some integrable characteristics and can reduce to it in some extreme conditions. We would like to point out that the Bäcklund transformation and corresponding KW theorem have been given for generalized models [4, 6] and [10]–[12].

Recently, a new generalization of GD hierarchy, called noncommutative GD (ncGD) and KP (ncKP) hierarchy, was introduced by P. Etingof, I.M. Gelfand and V.S. Retakh [13]. They found the noncommutative multi-soliton solution by means of the quasi-determinant. Kupershmidt's new book [14] developed a magnificent theory of the noncommutative KP hierarchy on noncommutative associative algebra. He succeeded in constructing the first hamiltonian structure and some important integrable characteristics of ncKP and ncGD hierarchy. However, the Bäcklund transformation for ncGD hierarchy is still unexplored. To understand deeply and entirely the integrable characteristics of a noncommutative system, it is worthwhile to study the Bäcklund transformation associated with the ncGD hierarchy. This is the first step to investigating the KW theorem of the ncGD and ncKP hierarchy.

The main purpose of this paper is to discuss the Bäcklund transformation of the ncGD hierarchy, and the Miura transformation between the ncGD and the ncMKdV hierarchy. After a brief review of ncGD hierarchy based on a n^{th} order differential operator with

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noncommutative coefficients as Lax operator, we show, in section 3, the ncMKdV hierarchy and the Miura transformation which result from the factorization of Lax operator for ncGD hierarchy. We will also show that the ncMKdV equations are invariant under cyclic permutation,and hence provide the Bäcklund transformation of the ncGD hierarchy. Our presentation follows closely the method of M. Adler [15]. The concluding remarks are given in section 4.

2. Noncommutative Gelfand-Dickey hierarchy

In this section we will follow [13] and [14] to give a brief review of the noncommutative Gelfand-Dickey hierarchy.

Let the differential algebra R be the algebra of polynomials in the non-commuting variables $u_i^{(j)}$:

$$R = F\left\langle u_i^{(j)} \right\rangle, i, j \in \mathbb{Z}_+ , \tag{2.1}$$

with the derivative ∂ acting on the generators of R by the rule

$$\partial\left(u_i^{(j)}\right) = u_i^{(j+1)} . \tag{2.2}$$

Here $u_i^{(j)} = \partial^j u_i / \partial x^j$. The field F will always be of characteristic zero. R is an associative (not necessary commutative) algebra with unity. Let L, called Lax operator for ncKP hierarchy,

$$L = \partial + u_0\partial^{-1} + u_1\partial^{-2} + u_2\partial^{-3} + \cdots , \tag{2.3}$$

be the formal Pseudo-differential operator whose coefficients belong to associative algebra R. The non-commutative KP hierarchy is defined by

$$\frac{\partial L}{\partial t_m} = [B_m, L] , \qquad B_m = L_+^m , \tag{2.4}$$

where for a pseudo-differential operator $M = \sum\limits_{i<\infty} X_i\partial^i$, and $M_+ = \sum\limits_{i\geq 0} X_i\partial^i$, $M_- = \sum\limits_{i<0} X_i\partial^i$. For brevity we will write L_+^m instead of $(L^m)_+$. It can be proven that the flows defined by equations (2.4) commute with each other.

The ncGD hierarchy can be reduced from the ncKP hierarchy by restricting its Lax operator to $L_-^n = 0$. Then

$$\frac{\partial M}{\partial t_m} = [M_+^{m/n}, M] , \qquad M = L^n = \partial^n + \tilde{u}_{n-2}\partial^{n-2} + \tilde{u}_{n-3}\partial^{n-3} + \cdots + \tilde{u}_1\partial + \tilde{u}_0 . \tag{2.5}$$

The flows in (2.5) are called noncommutative Gelfand-Dickey hierarchy. Among these flows, the flows corresponding to $m = nl, l \in \mathbb{Z}_+$, are trivial, but the other flows are nontrivial. For $n = 2$ the ncGD hierarchy is the noncommutative KdV(ncKdV) hierarchy. The first two nontrivial equations of this hierarchy are

$$u_{t_1} = u_x , \qquad u_{t_3} = \frac{1}{4}(u_{xxx} + 3uu_x + 3u_xu) . \tag{2.6}$$

The second equation is the ncKdV equation.

3. Bäcklund transformation of ncGD hierarchy

For convenience, we adjust notation of Lax operator and Lax equation for ncGD hierarchy as

$$\frac{\partial L}{\partial t_m} = [L_+^{m/n}, L] , \qquad L = \partial^n + \tilde{u}_{n-2}\partial^{n-2} + \tilde{u}_{n-3}\partial^{n-3} + \cdots + \tilde{u}_1\partial + \tilde{u}_0 . \tag{3.1}$$

The existence of the factorization for a differential operator L with order n is given by P. Etingof, I.M. Gelfand and V.S. Retakh [13].

Lemma 1. *[13] Differential operator given by eq.(3.1) admits factorization*

$$L = (\partial - v_{n-1})(\partial - v_{n-2}) \cdots (\partial - v_0) \tag{3.2}$$

if

$$\sum_{i=0}^{n-1} v_i = 0 .$$

Hence, we have the Miura transformation

$$\tilde{u}_i = Q(v_{n-1}, v_{n-2}, \cdots, v_1, v_0) , \tag{3.3}$$

where Q is differential polynomial of Miura variables(noncommutative) $v_i (i = 0, 1, \ldots, n-1)$. As we will see the v_i satisfy equations of ncMKdv hierarchy, so the Miura transformation is the relationship between the ncGD hierarchy and ncMKdV hierarchy. For example, $n = 2$,we have $L = \partial^2 + \tilde{u} = (\partial - v)(\partial + v)$, then $\tilde{u} = -v^2 + v_x$ as commutative case. The case of $n = 3$ will be given at the end of this section.

We can construct the parameterization of $v_i, i = 0, 1, \ldots, n-1$ as M.Adler [15]. Given the noncommutative functions $W = W(x) = (w_1(x), w_2(x), \ldots, w_{n-1}(x))$, ω primitive n^{th} root of unity, let

$$\begin{aligned}
W_0 &= W = (w_1(x), w_2(x), \ldots, w_{n-1}(x)), \\
W_1 &= \Omega W = (\omega w_1(x), \omega^2 w_2(x), \ldots, \omega^{n-1} w_{n-1}(x)), \\
W_j &= \Omega^j W = (\omega^j w_1(x), \omega^{2j} w_2(x), \ldots, \omega^{(n-1)j} w_{n-1}(x)), \\
A_0 &= A(W) = \partial + \sum_{i=1}^{n-1} w_i , \qquad A_j = A(W_j) \\
L_j &= A_{j-1}A_{j-2}\ldots A_0 A_n A_{n-1} A_{n-2} \ldots A_j , \qquad j = 0, 1, 2, \ldots, n-1 .
\end{aligned}$$

Lemma 2. *There exists at least one W such that $L_0 = L$.*

Proof. In order to prove Lemma 2, consider the parameterization of L_0

$$L_0 = \left(\partial + \sum_{k=1}^{n-1} \omega^{(n-1)k} w_k\right) \left(\partial + \sum_{k=1}^{n-1} \omega^{(n-2)k} w_k\right) \cdots \left(\partial + \sum_{k=1}^{n-1} \omega^k w_k\right) \left(\partial + \sum_{k=1}^{n-1} w_k\right)$$

and the factorization

$$L_0 = (\partial - v_{n-1})(\partial - v_{n-2}) \cdots (\partial - v_1)(\partial - v_0) .$$

We only need to show that there exists at least one set $\{w_i, i = 1, 2, \ldots, n-1\}$ such that $\sum\limits_{k=1}^{n-1} \omega^{jk} w_k = -v_j (j = 0, 1, 2, \ldots, n-1)$. This can be expressed by the following form of matrix equation,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega & \omega^2 & \cdots & \omega^{n-1} \\ \vdots & & & \vdots \\ \omega^{n-2} & \omega^{2(n-2)} & \ldots & \omega^{(n-1)(n-2)} \\ \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)^2} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n-1} \end{pmatrix} = - \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{pmatrix} \tag{3.4}$$

Note that the matrix in the left hand side of equation above is not a square matrix but has dimensions $n \times (n-1)$. In general this is an over-determined system. Nevertheless there exists one set $\{w_i, i = 1, 2, \ldots, n-1\}$ which is a solution of the first $n-1$ equations in eq. (3.4) because the corresponding $(n-1)\times(n-1)$ sub-matrix is non-singular. Furthermore, the last equation in eq. (3.4) is automatically satisfied

$$\begin{aligned} \sum_{i=0}^{n-2} v_i - \sum_{k=1}^{n-1} \omega^{k(n-1)} w_k &= -\sum_{i=0}^{n-2}\sum_{k=1}^{n-1} \omega^{ik} w_k - \sum_{k=1}^{n-1} \omega^{k(n-1)} w_k \\ &= -\sum_{i=0}^{n-1}\sum_{k=1}^{n-1} \omega^{ik} w_k = \sum_{k=1}^{n-1} \left(\sum_{i=0}^{n-1} \omega^{ik} \right) w_k = 0 \end{aligned}$$

because $\sum\limits_{i=0}^{n-1} \omega^{ik} = 0$. This completes the proof of the Lemma. □

To define the ncMKdV hierarchy reasonably, we need to analyse the properties of L_j. Note $L = L_0(W)$, and its differential part is $P = P(W) = P_0(W) = (L_0)_+^{m/n} = L_+^{m/n}$. L_j is a differential operator with order n and its second highest term equals zero. Obviously, we have $L_j(W) = L_0(W_j)$, and the corresponding differential part $P_j = P(W_j) = (L_j)_+^{m/n}$. We define the operator

$$B_j = B_0(W_j) = P_{j+1}A_j - A_j P_j \, , \qquad j = 0, 1, \cdots, n-1 \, , \tag{3.5}$$

$$\tilde{L}_j = \sum_{s=0}^{n-1} (A_{j-1}A_{j-2} \cdots A_{s+1} B_s A_{s-1} \cdots A_j) \, . \tag{3.6}$$

Lemma 3.

$$\tilde{L}_j = [P_j, L_j] \tag{3.7}$$

and the order of $[P_j, L_j]$ *is* $n-2$.

Proof. By direct computation,

$$\begin{aligned} \tilde{L}_j &= \sum_{s=0}^{n-1} (A_{j-1}A_{j-2} \cdots A_{s+1} B_s A_{s-1} \cdots A_j) \\ &= \sum_{s=0}^{n-1} ((A_{j-1}A_{j-2} \cdots A_{s+1} P_{s+1} A_s A_{s-1} \cdots A_j) - (A_{j-1}A_{j-2} \cdots A_{s+1} A_s P_s A_{s-1} \cdots A_j)) \\ &= P_j L_j - L_j P_j = [P_j, L_j] \, . \end{aligned}$$

□

Lemma 4. *Operators $B_j = P_{j+1}A_j - A_jP_j$ are of order zero.*

Proof. Let the order of operator B_j be $t > 0$, and $B_0(W) = b(W)\partial^t + \hat{b}(W)\partial^{t-1} + \cdots$. Then from the definition of $\tilde{L}_0$ and B_s, we deduce

$$
\begin{aligned}
\tilde{L}_0 &= \sum_{s=0}^{n-1}(A_{n-1}A_{n-2}\cdots A_{s+1}B_sA_{s-1}\cdots A_0) \\
&= \sum_{s=0}^{n-1}(\partial - v_{n-1})\cdots(\partial - v_{s+1})(b(W_s)\partial^t + \cdots)(\partial - v_{s-1})\cdots(\partial - v_0) \\
&= \left(\sum_{s=0}^{n-1} b(W_s)\right)\partial^{n+t-1} + E_0\partial^{n+t-2} + \text{lower order terms}\,.
\end{aligned}
$$

But $\tilde{L}_0$ is an $n-2$ order operator, hence at least the two highest terms must cancel out. This implies

$$\sum_{s=0}^{n-1} b(W_s) = 0\,, \tag{3.8}$$

$$E_0 = \sum_{s=0}^{n-1}(n-s-1)b_x(W_s) + \sum_{s=0}^{n-1}\hat{b}(W_s) - \sum_{s=0}^{n-1}\left(\sum_{r<s} v_r b(W_s) + \sum_{r>s} b(W_s)v_r\right) = 0\,. \tag{3.9}$$

If we consider $\tilde{L}_k$ obtained by the cyclic permutation $W_0 \to W_k$ from $\tilde{L}_0$, the more general identity for the second highest term is

$$E_k = \sum_{s=0}^{n-1}(n-s-1)b_x(W_{s+k}) + \sum_{s=0}^{n-1}\hat{b}(W_{s+k}) - \sum_{s=0}^{n-1}\left(\sum_{r<s} v_{r+k}b(W_{s+k}) + \sum_{r>s} b(W_{s+k})v_{r+k}\right) = 0\,. \tag{3.10}$$

As in the commutative case, it is very clear that the $\sum_{s=0}^{n-1}(n-1)b_x(W_{s+k})$ and $\sum_{s=0}^{n-1}\hat{b}(W_{s+k})$ in (3.10) are independent of k. Note that there is distinct difference between the noncommutative and commutative cases as can be seen by considering the last term in (3.10). It is not obvious that this term is independent of k. It can be proved, however, by Lemma 5. So the remaining term $\sum_{s=0}^{n-1} sb_x(W_{s+k})$ in (3.10) does not depend on k, and so is $\sum_{s=0}^{n-1} sb(W_{s+k})$. Assume $k = 1$ without loss of the generality, and we have

$$
\begin{aligned}
0 &= \sum_{s=0}^{n-1} sb(W_s) - \sum_{s=0}^{n-1} sb(W_{s+1}) \\
&= (\, b(W_1) + 2b(W_2) + \cdots + (n-2)b(W_{n-2}) + (n-1)b(W_{n-1})\,) - \\
&\quad - (\, b(W_2) + 2b(W_3) + \cdots + (n-2)b(W_{n-1}) + (n-1)b(W_0)\,) \\
&= b(W_1) + b(W_2) + \cdots + b(W_{n-2}) + b(W_{n-1}) - (n-1)b(W_0) = -nb(W_0)
\end{aligned}
$$

and so $b(W_0) = 0$. We used the $\sum_{s=0}^{n-1} b(W_s) = 0$ in the last step. Similarly, $b(W_j) = 0$. This contradicts our assumption. Thus $B_j(W)$ are of order zero. □

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Lemma 5. *The third part in (3.10) is independent of k.*

Proof. We calculate directly the third part of the second highest term in $\tilde{L}_k = \sum_{s=0}^{n-1}(A_{k-1} \times A_{k-2} \cdots A_{s+1} B_s A_{s-1} \cdots A_k)$ denoted by $\tilde{L}_k(3)$ instead of the third part in equation (3.10). In$\tilde{L}_k(3)$, B_s contributes the highest term $b(W_s)\partial^t$, one of the A_i offers its the second highest term $-v_i$ and all the others of $A_j (j \neq i)$ offer their highest term ∂. So we can get

$$
\begin{aligned}
\tilde{L}_0(3) &= \tilde{L}_0(3, b_0) + \tilde{L}_0(3, b_1) + \cdots + \tilde{L}_0(3, b_k) + \cdots + \tilde{L}_0(3, b_{n-1}) \\
&= -(v_{n-1} + v_{n-2} + \cdots + v_2 + v_1)b_0 - \\
&\quad -(v_{n-1} + v_{n-2} + \cdots + v_2)b_1 - b_1 v_0 - \\
&\quad \vdots \\
&\quad -(v_{n-1} + v_{n-2} + \cdots + v_{k-1})b_{k-2} - b_{k-2}(v_{k-3} + \cdots + v_1 + v_0) - \\
&\quad -(v_{n-1} + v_{n-2} + \cdots + v_k)b_{k-1} - b_{k-1}(v_{k-2} + \cdots + v_1 + v_0) - \\
&\quad -(v_{n-1} + v_{n-2} + \cdots + v_{k+1})b_k - b_k(v_{k-1} + v_{k-2} + \cdots + v_1 + v_0) - \\
&\quad -(v_{n-1} + v_{n-2} + \cdots + v_{k+2})b_{k+1} - b_{k+1}(v_k + v_{k-1} + \cdots + v_1 + v_0) - \\
&\quad \vdots \\
&\quad - v_{n-1}b_{n-2} - b_{n-2}(v_{n-3} + v_{n-4} + \cdots + v_1 + v_0) - \\
&\quad - b_{n-1}(v_{n-2} + v_{n-3} + \cdots + v_1 + v_0)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{L}_k(3) &= \tilde{L}_k(3, b_k) + \tilde{L}_k(3, b_{k+1}) + \cdots + \tilde{L}_k(3, b_{n-1}) + \tilde{L}_k(3, b_0) + \cdots + \tilde{L}_k(3, b_{k-1}) \\
&= -(v_{k-1} + v_{k-2} + \cdots + v_1 + v_0 + v_{n-1} + v_{n-2} + \cdots + v_{k+1})b_k - \\
&\quad -(v_{k-1} + v_{k-2} + \cdots + v_1 + v_0 + v_{n-1} + v_{n-2} + \cdots + v_{k+2})b_{k+1} - b_{k+1}v_k - \\
&\quad \vdots \\
&\quad -(v_{k-1} + v_{k-2} + \cdots + v_1 + v_0 + v_{n-1})b_{n-2} - b_{n-2}(v_{n-3} + v_{n-4} + \cdots + v_k) - \\
&\quad -(v_{k-1} + v_{k-2} + \cdots + v_1 + v_0)b_{n-1} - b_{n-1}(v_{n-2} + v_{n-3} + \cdots + v_k) - \\
&\quad -(v_{k-1} + v_{k-2} + \cdots + v_1)b_0 - b_0(v_{n-1} + v_{n-2} + \cdots + v_k) - \\
&\quad -(v_{k-1} + v_{k-2} + \cdots + v_2)b_1 - b_1(v_0 + v_{n-1} + v_{n-2} + \cdots + v_k) - \\
&\quad \vdots \\
&\quad - v_{k-1}b_{k-2} - b_{k-2}(v_{k-3} + \cdots + v_1 + v_0 + v_{n-1} + v_{n-2} + \cdots + v_k) - \\
&\quad - b_{k-1}(v_{k-2} + \cdots + v_1 + v_0 + v_{n-1} + v_{n-2} + \cdots + v_k)\,.
\end{aligned}
$$

In the above, $\tilde{L}_k(3, b_s)$ denotes one term in $\tilde{L}_k(3)$ which contains $b(W_s)$. It is a useful way to compare terms involving $b(W_i)(i = 0, 1, 2, \ldots n-1)$ in $\tilde{L}_0$ and $\tilde{L}_k$,respectively. By direct observation from explicit forms of $\tilde{L}_0(3)$ and $\tilde{L}_k(3)$ above, we obtain

$$
\left(\tilde{L}_0 - \tilde{L}_k\right)(3, b_s) = -[b(W_s), v_{k-1} + v_{k-2} + \cdots + v_0]\,, \quad (s = k, k+1, \ldots, n-1), \tag{3.11}
$$

$$
\left(\tilde{L}_0 - \tilde{L}_k\right)(3, b_s) = -[v_{n-1} + v_{n-2} + \cdots + v_k, b(W_s)]\,, \quad (s = 0, 1, 2, \ldots, k-1)\,. \tag{3.12}
$$

Hence it implies

$$\begin{aligned}\tilde{L}_0(3) - \tilde{L}_k(3) \;=\; & -[b(W_k) + b(W_{k+1}) + \cdots + b(W_{n-1}), v_{k-1} + v_{k-2} + \cdots + v_0] + \\ & + [b(W_0) + b(W_1) + \cdots + b(W_{k-1}), v_{n-1} + v_{n-2} + \cdots + v_k]\,.\end{aligned}$$

Note that the right hand side of above equation does indeed equal zero because of $\sum_{s=0}^{n-1} b(W_s)$ $= 0$ and $\sum_{s=0}^{n-1} v_s = 0$. This proves Lemma 5. □

We are now in a position to discuss the Bäcklund transformation of the ncGD hierarchy. Here, we define ncMKdV equations as

$$\frac{dA_j}{dt_m} = -\frac{dv_j}{dt_m} \equiv B_j(W) = P_{j+1}A_j - A_jP_j, j = 0, 1, 2, \ldots, n-1\,, \tag{3.13}$$

for the variables $t_m(m = 1, 2, \ldots, n-1)$. We still need to think whether the definition of $\frac{dv_j}{dt_m}$ is reasonable from the point of view of Lax equation eq. (3.1) although B_j has been proved to be a differential operator of order zero.

Lemma 6. *(i) The flows defined in eq.(3.13)are compatible with Lax equation eq.(3.1). (ii) The ncMKdV equations are invariant under cyclic permutation* Ω:$v_0 \to v_1, v_1 \to v_2, v_2 \to v_3, \ldots, v_{n-1} \to v_0$.

Proof. (i)

$$\frac{\partial L}{\partial t_m} = \sum_{j=0}^{n-1} A_{n-1}A_{n-2}\cdots\frac{\partial A_j}{\partial t_m}A_{j-1}\cdots A_0\,.$$

Using the equation (3.13)and the proof of the Lemma 3, it is follows that

$$\begin{aligned}\frac{\partial L}{\partial t_m} &= \sum_{j=0}^{n-1} A_{n-1}A_{n-2}\cdots B_jA_{j-1}\cdots A_0 \\ &= \tilde{L}_0 = [P, L]\,.\end{aligned}$$

Here $P = L_+^{m/n}$.

(ii) It is reduced directly from its flows $\frac{dv_j}{dt_m} \equiv -B_j(W)$. □

Then we can present the following results concerning the Bäcklund transformation of ncGD hierarchy.

Theorem 1. *The cyclic permutation* Ω *generates the Bäcklund transformation of ncGD hierarchy.*

Proof. We just need to check that the Lax equation (3.1) is invariant under the cyclic permutation as follows. With the help of the definition of L_j, equation (3.6) and Lemma 3, we have

$$
\begin{aligned}
\frac{\partial L_j}{\partial t_m} &= \frac{\partial}{\partial t_m} A_{j-1}A_{j-2}\cdots A_0 A_{n-1}A_{n-2}\cdots A_j \\
&= \sum_{s=0}^{n-1}\left(A_{j-1}A_{j-2}\cdots A_{s+1}\frac{\partial A_s}{\partial t_m}A_{s-1}\cdots A_j\right) \\
&= \sum_{s=0}^{n-1}\left(A_{j-1}A_{j-2}\cdots A_{s+1}B_s(W)A_{s-1}\cdots A_j\right) \\
&= \tilde{L}_j = [P_j, L_j]\,.
\end{aligned}
$$

This conclues the proof of the theorem. □

In fact we can regard the process to Bäcklund transformation discussed above as following four steps:

- factorize L into n first order operators $\partial - v_i$(Lemma 1),
- parameterize v_i so that $\sum_{k=0}^{n-1} v_k = 0$(Lemma 2),
- define equation of $\frac{\partial v_i}{\partial t} = B(v_0, v_1, \ldots, v_{n-1})$(Lemma 6 with help of Lemma 3–5)
- prove $\frac{\partial v_i}{\partial t}$ to be invariant under cyclic permutation(Lemma 6).

At last, we would like to present one example to illustrate this approach. Consider the noncommutative Gefand-Dickey hierarchy of order 3. Its factorization is

$$
L = \partial^3 + u_1\partial + u_0 = (\partial - v_2)(\partial - v_1)(\partial - v_0)\,, \tag{3.14}
$$

and the Miura transformation

$$
u_1 = -2v_{0x} - v_{1x} - v_0^2 - v_0v_1 - v_1^2\,, \tag{3.15}
$$

$$
u_0 = -v_{0xx} - v_0v_{0x} + v_{1x}v_0 - (v_1 + v_0)v_1v_0\,. \tag{3.16}
$$

Their parameterizations are

$$
\begin{aligned}
v_0 &= w_1 + w_2\,, \\
v_1 &= w_1e^{i2\pi/3} + w_2e^{i4\pi/3}\,, \\
v_2 &= w_1e^{i4\pi/3} + w_2e^{i8\pi/3}\,,
\end{aligned}
$$

and $\omega = e^{i2\pi/3}$. So we can do cyclic permutation Ω:$v_0 \to v_1, v_1 \to v_2, v_2 \to v_0$, and then get

$$
L_1 = \partial^3 + \tilde{u}_1\partial + \tilde{u}_0 = (\partial - v_0)(\partial - v_2)(\partial - v_1)\,, \tag{3.17}
$$

$$
\tilde{u}_1 = v_{0x} - v_{1x} - v_1^2 - v_0(v_1 + v_0)\,, \tag{3.18}
$$

$$
\tilde{u}_0 = -v_{1xx} - v_1v_{1x} - v_{1x}v_1 - v_{0x}v_1 - v_0^2v_1 - v_0v_1^2\,. \tag{3.19}
$$

According to the Theorem above, the transformation: $(u_1, u_0) \longrightarrow (\tilde{u}_1, \tilde{u}_0)$, is one Bäcklund transformation for noncommutative Gelfand-Dickey hierarchy. A non-trivial factorization for a third order differential operator was done by Fordy and Gibbons in [20]. We will discuss more details of its non-commutative analogue in another paper.

4. Concluding remarks

In this paper we define the ncMKdV hierarchy (3.13) and the Miura transformation (3.3) by factorizing the Lax operator (3.1) of the ncGD hierarchy. Based on the invariance of the ncMKdV under cyclic permutation, the Baĉklund transformation $L \to L_j$ of ncGD hierarchy is given as the Theorem.

It is well known that the factorization of the Lax operator for the usual GD hierarchy leads to a decomposition of its second Poisson bracket, called KW theorem [2, 16]. This theorem has been generalized to KP hierarchy [10] and the supersymmetric KP hierarchy [11]. So we are interested in asking whether there exists non-commutative analogue of the KW theorem for the noncommutative GD and KP hierarchy. We know that the hamiltonian theory [14] of the noncommutative integrable system is not at all easy. On the other hand, it is an interesting problem to solve the ncGD and ncKP by gauge transformation, because the gauge transformation is an efficient way to solve KP and constraint KP [17]–[19]. We hope we can answer these questions in the near future.

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